

# Octonion X-product and Octonion $E_8$ Lattices

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## Abstract

In this episode, it is shown how the octonion X-product is related to  $E_8$  lattices, integral domains, sphere fibrations, and other neat stuff.

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\*supported by the solidity of the planet earth, without which none of this work would have been possible.

## 1. Introduction.

Let  $\mathbf{O}$  be the octonion algebra [1], an 8-dimensional real division algebra, both noncommutative and nonassociative, and the last in the finite sequence of real division algebras including the reals,  $\mathbf{R}$ , complexes,  $\mathbf{C}$ , and quaternions,  $\mathbf{Q}$ . Let  $e_a$ ,  $a = 0, 1, \dots, 7$ , be a basis for  $\mathbf{O}$ , with

$$e_0 = 1, \quad (1)$$

the identity, and

$$(e_a)^2 = -1, \quad a \in \{1, \dots, 7\}. \quad (2)$$

These latter elements also anticommute:

$$e_a e_b = -e_b e_a, \quad a \neq b \in \{1, \dots, 7\}. \quad (3)$$

Finally, we choose an octonion multiplication whose quaternionic triples are determined by the cyclic product rule,

$$e_a e_{a+1} = e_{a+5}, \quad a \in \{1, \dots, 7\}, \quad (4)$$

where the indices in (4) are from 1 to 7, modulo 7 (and in particular we will set  $7=7 \bmod 7$  to avoid confusing  $e_0$  with  $e_7$ ). Given (4), the following useful property is also satisfied:

$$e_a e_b = \pm e_c \implies e_{2a} e_{2b} = \pm e_{2c}, \quad a, b, c \in \{1, \dots, 7\} \quad (5)$$

(this is the index doubling property; because of (4) and (5), proofs of many octonion properties can be done very generally by proving the property in one example only).

Let  $X \in \mathbf{O}$  be a unit element. That is,

$$\|X\|^2 = XX^\dagger = (X^0 + \sum_{a=1}^7 X^a e_a)(X^0 - \sum_{a=1}^7 X^a e_a) = \sum_{a=0}^7 (X^a)^2 = 1. \quad (6)$$

So,

$$X \in S^7, \quad (7)$$

the 7-sphere. Because  $\mathbf{O}$  is nonassociative, if  $X \in S^7$ , and  $A, B \in \mathbf{O}$  are two other elements, then

$$A \circ_X B \equiv (AX)(X^\dagger B) \neq AB \quad (8)$$

in general. However, if we fix  $X$ , then  $\mathbf{O}_X$ , which denotes  $\mathbf{O}$  with its original product replaced by the so-called X-product (8) (see [2], and also [1][3]), is yet another copy of the octonions, isomorphic to the starting copy (or any other copy).

In [3] I showed that if  $X \in \Xi_0 \cup \Xi_1 \cup \Xi_2 \cup \Xi_3$ , where

$$\begin{aligned}\Xi_0 &= \{\pm e_a\}, \\ \Xi_1 &= \{(\pm e_a \pm e_b)/\sqrt{2} : a, b \text{ distinct}\}, \\ \Xi_2 &= \{(\pm e_a \pm e_b \pm e_c \pm e_d)/2 : a, b, c, d \text{ distinct, } e_a(e_b(e_c e_d)) = \pm 1\}, \\ \Xi_3 &= \{(\sum_{a=0}^7 \pm e_a)/\sqrt{8} : \text{odd number of } +\text{'s}\}, \\ &\quad a, b, c, d \in \{0, \dots, 7\},\end{aligned}\tag{9}$$

then for all  $a, b \in \{0, \dots, 7\}$ , there is some  $c \in \{0, \dots, 7\}$  such that

$$e_a \circ_X e_b = \pm e_c \tag{10}$$

(in [3] the superscript +5 was affixed to the sets  $\Xi_m$  to indicate that the starting multiplication was that determined by (4), but as I am using only this one starting multiplication here, I will dispense with the superscripts). That is, in this case  $\mathbf{O}_X$  can be obtained from  $\mathbf{O}$  by a rearrangement of the indices in  $\{1, \dots, 7\}$  [3].

## 2. $E_8$ Lattices and Integral Domains.

The 240 elements of  $\Xi_0 \cup \Xi_2$  are the nearest neighbors (first shell) to the origin of an  $E_8$  lattice (so are the 240 elements of  $\Xi_1 \cup \Xi_3$  (see [4][5])). Define

$$\mathcal{E}_8^h = G^h[\Xi_0 \cup \Xi_2], \quad h = 1, \dots, 7, \tag{11}$$

where  $G^h$  is the  $O(8)$  reflection taking  $e_0 \longleftrightarrow e_h$ . These 7 sets are nearest neighbor points for 7 different  $E_8$  lattices, but in this case it is well known [4][5] that the 240 points of  $\mathcal{E}_8^h$ , for each  $h = 1, \dots, 7$ , close under multiplication (however, because of nonassociativity they do not form a finite group). One may also think of  $\mathcal{E}_8^h$  as being the unital elements of a noncommutative and nonassociative integral domain.

It should be fairly obvious that if

$$X \in \mathcal{E}_8^h,$$

then  $\mathcal{E}_{8X}^h$ , the X-product variant of  $\mathcal{E}_8^h$ , is also closed under its multiplication, since for all  $A, B \in \mathcal{E}_8^h$ ,

$$AX \in \mathcal{E}_8^h \quad \text{and} \quad X^\dagger B \in \mathcal{E}_8^h \implies (AX)(X^\dagger B) \in \mathcal{E}_8^h.$$

(Clearly if  $X \in \mathcal{E}_8^h$ , then  $X^\dagger \in \mathcal{E}_8^h$ .) However, only if  $X$  is also an element of  $\Xi_0 \cup \Xi_2$  will the resulting X-product also satisfy (10). For example,

$$\mathcal{E}_8^7 \cap \Xi_0 = \Xi_0,$$

and

$$\begin{aligned} \mathcal{E}_8^7 \cap \Xi_2 = & \{(\pm 1 \pm e_1 \pm e_5 \pm e_7)/2, (\pm e_2 \pm e_3 \pm e_4 \pm e_6)/2, \\ & (\pm 1 \pm e_2 \pm e_3 \pm e_7)/2, (\pm e_4 \pm e_6 \pm e_1 \pm e_5)/2, \\ & (\pm 1 \pm e_4 \pm e_6 \pm e_7)/2, (\pm e_1 \pm e_5 \pm e_1 \pm e_5)/2\}. \end{aligned}$$

Since  $\pm X$  results in the same X-product, there are 8 X-product variants  $\mathcal{E}_{8X}^7$  arising from  $\mathcal{E}_8^7 \cap \Xi_0$ , and  $(6 \times 16)/2 = 48$  arising from  $\mathcal{E}_8^7 \cap \Xi_2$ . So there are 56 X-product variants of  $\mathcal{E}_8^7$  (and by virtue of index cycling, all the  $\mathcal{E}_8^h$ ) that satisfy (10), and close under the new X-product. (There are clearly other X-product variants that close but do not satisfy (10), that is, for which the resulting multiplication is not the result of a simple index rearrangement [3].)

### 3. $\mathcal{E}_8^0 \equiv \Xi_0 \cup \Xi_2$ is Closed.

Actually,

$$\mathcal{E}_8^0 \equiv \Xi_0 \cup \Xi_2$$

is not closed under the multiplication we started with. For example,

$$(1 + e_1 + e_2 + e_6)/2, (1 + e_1 + e_3 + e_4)/2 \in \mathcal{E}_8^0,$$

but

$$(1 + e_1 + e_2 + e_6)(1 + e_1 + e_3 + e_4)/4 = (e_1 + e_2 + e_4 + e_5)/2 \notin \mathcal{E}_8^0,$$

because

$$e_1(e_2(e_4e_5)) = e_1(e_2e_2) = -e_1 \neq \pm 1.$$

However, take a look at the  $\mathcal{E}_8^h, h = 1, \dots, 7$ . These satisfy

$$\mathcal{E}_8^h = \Xi_0 \cup \Xi_2^h, \tag{12}$$

where

$$\begin{aligned} \Xi_2^h = & \{(\pm e_a \pm e_b \pm e_c \pm e_d)/2 : a, b, c, d \text{ distinct}, \\ & e_a(e_b(e_c e_d)) = \pm e_h \text{ if exactly one of } a, b, c, d = 0 \text{ or } h, \\ & e_a(e_b(e_c e_d)) = \pm 1 \text{ otherwise } \}. \end{aligned} \tag{13}$$

As it turns out, we can achieve very much the same thing on  $\mathcal{E}_8^0$  using the X-product. Consider

$$X = (1 + e_7)/\sqrt{2} \in \Xi_1. \tag{14}$$

Let

$$A = (\pm e_a \pm e_b \pm e_c \pm e_d)/2 \in \Xi_2, \quad (15)$$

so

$$e_a(e_b(e_c e_d)) = \pm 1. \quad (16)$$

If, however, we modify the product, using the X-product (8) with  $X$  given in (14), then the bits of  $A$  in (15) satisfying (16) also satisfy

$$\begin{aligned} e_a \circ_X (e_b \circ_X (e_c \circ_X e_d)) &= \pm e_7 \\ \text{if exactly one of } a, b, c, d &= 0 \text{ or } h = 7, \\ e_a(e_b(e_c e_d)) &= \pm 1 \text{ otherwise.} \end{aligned} \quad (17)$$

In other words (see (13)),  $\mathcal{E}_8^0$  is closed under this particular X-product. Note, for example, that

$$(1 + e_1 + e_2 + e_6) \circ_X (1 + e_1 + e_3 + e_4)/4 = (e_2 + e_3 + e_4 + e_6)/2 \in \Xi_2, \quad (18)$$

since

$$e_2(e_3(e_4 e_6)) = e_2(e_3 e_7) = e_2(e_2) = -1$$

(don't forget,  $\Xi_2$  is defined in terms of the beginning product, not the X-product variant). Our principal result is then the following:

$$\boxed{\mathcal{E}_{8X}^0 \text{ is X-product closed if } X \in \Xi_1.} \quad (19)$$

Given that modulo sign change  $\Xi_1$  has 56 elements, there are also therefore 56 X-products variants  $\mathcal{E}_{8X}^0$  of  $\mathcal{E}_8^0$  that are closed under multiplication, and from which we may define integral domains.

## 4. Sphere Fibrations to Lattice Fibrations.

Let

$$X = X^0 + X^1 e_1 + X^2 e_2 + X^3 e_3 + X^4 e_4 + X^5 e_5 + X^6 e_6 + X^7 e_7 \in S^7.$$

Then

$$\begin{aligned}
e_1 \circ_X e_2 &= \\
&((X^0)^2 + (X^1)^2 + (X^2)^2 + (X^6)^2 - (X^3)^2 - (X^4)^2 - (X^5)^2 - (X^7)^2)e_6 \\
&+ 2(X^0X^5 + X^1X^7 - X^2X^4 + X^3X^6)e_3 \\
&+ 2(-X^0X^7 + X^1X^5 + X^2X^3 + X^4X^6)e_4 \\
&+ 2(-X^0X^3 - X^1X^4 - X^2X^7 + X^5X^6)e_5 \\
&+ 2(X^0X^4 - X^1X^3 + X^2X^5 + X^7X^6)e_7 \\
&= Y = Y^6e_6 + Y^3e_3 + Y^4e_4 + Y^5e_5 + Y^7e_7
\end{aligned} \tag{20}$$

(see [1][3]) defines a map from  $S^7 \longrightarrow S^4$ . That is,

$$(Y^6)^2 + (Y^3)^2 + (Y^4)^2 + (Y^5)^2 + (Y^7)^2 = 1. \tag{21}$$

Therefore, relative to this X-product, the set  $\{e_1, e_2, Y\}$  is a quaternionic triple. This implies that the set of all

$$U = \exp(\theta^1 e_1 + \theta^2 e_2 + \theta^3 Y) \tag{22}$$

is just  $SU(2) \simeq S^3$ , and

$$e_1 \circ_{(UX)} e_2 = (e_1 \circ_X U) \circ_X (U^\dagger \circ_X e_2) = e_1 \circ_X e_2 = Y \tag{23}$$

(see [3]), since the  $U$  and  $U^\dagger$  cancel each other out because these two elements are part of the quaternionic subalgebra of  $\mathbf{O}_X$  generated by  $e_1$  and  $e_2$ . Therefore,

$$\{UX : U = \exp(\theta^1 e_1 + \theta^2 e_2 + \theta^3 Y)\} \simeq S^3 \tag{24}$$

is the  $S^3$  fibre over  $Y \in S^4$  in the exact sequence

$$S^3 \longrightarrow S^7 \longrightarrow S^4, \tag{25}$$

implicit in (20-24). (Clearly the map (20) could be replaced by

$$X \longrightarrow e_a \circ_X e_b, \quad a \neq b \in \{1, \dots, 7\};$$

many other possibilities exist, which I will leave to the reader to explore.)

All of this translates to lattices, the shells of which are discrete versions of spheres. In particular, let

$$X \in \mathcal{E}_8^0 \equiv \Xi_0 \cup \Xi_2 \subset S^7. \tag{26}$$

Consider the map

$$X \longrightarrow e_1 \circ_X e_2. \quad (27)$$

Because of (10), the image of this map is the ten element set

$$\mathcal{Z}^5 \equiv \{\pm e_6, \pm e_3, \pm e_4, \pm e_5, \pm e_7\} \subset S^4, \quad (28)$$

which is the inner shell of the 5-dimensional cubic lattice,  $\mathbf{Z}^5$  (see [5]). Consider the fibre of elements of  $\mathcal{E}_8^0$  mapping to  $e_6 \in \mathcal{Z}^5$ , which is

$$\mathcal{D}_4 \equiv \{\pm 1, \pm e_1, \pm e_2, \pm e_6\} \cup \{(\pm 1 \pm e_1 \pm e_2 \pm e_6)/2\} \subset S^3 \longrightarrow \{e_6\}, \quad (29)$$

which is the inner shell of a 24-dimensional  $D_4$  lattice and integral domain (see [4][5]). Generalizing further from (25), we have an exact sequence

$$\mathcal{D}_4 \subset S^3 \longrightarrow \mathcal{E}_8^0 \subset S^7 \longrightarrow \mathcal{Z}^5 \subset S^4. \quad (30)$$

- NOTE:  $10 \times 24 = 240$ , which I leave to the reader to prove.

## 5. Conclusion.

The motivation for this work is curiosity fired by the beauty of the mathematics. I have only scratched the surface of this vast interconnected mathematical realm, and many of its connections I will never see. But I share a profound belief that the design of our physical reality is intimately linked to this web of mathematical notions [1], and this is my way of gaining a better understanding of the web and its power.

## References

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